

MIT OCW Real Analysis

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Week 1

Exercise 0.3.6

Prove:

- a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

a) In order to prove this equivalence, we have to prove the implication both ways. We use two lemmas for this.

Lemma 1.1 — $A \cap (B \cup C) \implies (A \cap B) \cup (A \cap C)$

Let $x \in A \cap (B \cup C)$. By the definition of set intersection, $x \in A$ and $x \in B \cup C$. By the definition of set union, $x \in A$ and ($x \in B$ or $x \in C$). From propositional logic we know that for propositions P, Q and R the following holds: $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$. So, substituting for this particular case yields ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). Using the definition of set intersection again gets $x \in A \cap B$ or $x \in A \cap C$. Using the definition of set union again gives $x \in (A \cap B) \cup (A \cap C)$. $\square$

Lemma 1.2 — $(A \cap B) \cup (A \cap C) \implies A \cap (B \cup C)$

Let $x \in (A \cap B) \cup (A \cap C)$. By the definition of set union, $x \in (A \cap B)$ or $x \in (A \cap C)$. By the definition of set intersection, ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). Using the same propositional logical equivalence as in Lemma 1.1, this gives $x \in A$ and ($x \in B$ or $x \in C$). Wrapping up, we use the definition of set union to get $x \in A$ and $x \in B \cup C$ and the definition of intersection to get $x \in A \cap (B \cup C)$. $\square$

Using Lemma 1.1 and 1.2, we get the desired equivalence of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. $\square$

b) This proof is so similar to a) that it feels like a waste of time and will therefore be left to the reader.

Exercise 0.3.11

Prove by induction that $n < 2^n$ for all $n \in \mathbb{N}$.

For this proof we will use induction. For this, we have to prove the base case, i.e. $n = 1$, and the inductive step, $n < 2^n \implies n + 1 < 2^{n+1}$.

First, let's prove the base case. When $n = 1$, we get $1 < 2^1$, which is certainly true.

Then, for the inductive step. We assume that the proposition holds for any $m \in \mathbb{N}$. So, $m < 2^m$. Multiplying both sides with 2 gives $2m < 2^{m+1}$. Since $m \geq 1$, $m + 1 < 2m$, and thus $m + 1 < 2m < 2^{m+1}$.

Since both the base case and inductive step hold, we can close the induction, proving the proposition. $\square$

Exercise 0.3.12

Show that for a finite set A of cardinality n , the cardinality of $\mathcal{P}(A)$ is 2^n .

The power set of a set A , $\mathcal{P}(A)$, is defined as the set of all possible subsets of A . This is very similar to an inclusion/exclusion problem. It is built up by all the possible combinations of the different elements being either inside a certain subset or not. For all possible subsets of A , we have that for every element $x \in A$ there are 2 possibilities, either x is in the subset or it isn't. This means that for every additional element, the number of subsets increases by a factor of 2, with a minimum of 1, in case of $A = \emptyset$. We will prove this formally now, using induction.

For this, the base case is a set of 1 element (but the theorem also holds for the empty set, where $n = 0$). Let us assume that $A := \{\pi\}$. Then the cardinality of $\mathcal{P}(A)$ is 2^1 , with $\mathcal{P}(A) = \{\emptyset, \{\pi\}\}$.

For the inductive step, we assume that for any set B of cardinality m , the cardinality of the power set of B is 2^m . Then, we will add an element $x \notin B$ to B to increase its cardinality by 1, to $m + 1$, creating a new set C . Note that all the possible subsets of B are still viable subsets of C , since $B \subset C$. In order to create the new subsets, we can simply keep all the subsets of B , duplicate them and take the union with the new element x , so now we also have all combinations of the old sets with possibly x being in them. Since this doubles the number of subsets, the cardinality of $\mathcal{P}(C)$ is 2^{m+1} .

Both the base case and the inductive step hold, which closes the induction and proves the proposition. $\square$

Exercise 0.3.15

Prove that $n^3 + 5n$ is divisible by 6 for all $n \in \mathbb{N}$.

In order to prove this proposition, we will use induction. To do this, we need to prove the following lemma, of which we will see the usefulness later:

Lemma 4.1 — $3n^2 + 3n + 6$ is divisible by 6 for all $n \in \mathbb{N}$.

This lemma we will also prove by induction. For this, we prove the base case and the inductive step. First, for the base case we have $n = 1$, yielding $3 \cdot 1^2 + 3 \cdot 1 + 6 = 12$, which is divisibly by 6.

Then, for the inductive step we assume that the lemma holds for a certain $m \in \mathbb{N}$. So, $3m^2 + 3m + 6$ is divisible by 6. Substituting m with $m + 1$ gives $3(m + 1)^2 + 3(m + 1) + 6$, which can be expanded to $3m^2 + 9m + 12$. Rewriting this with our assumption in mind gives the following: $(3m^2 + 3m + 6) + (6m + 6)$. We know from our assumption that the first part is divisible by 6, and since $m \in \mathbb{N}$, $6m + 6$ is also divisible by 6, and so the whole expression is as well. $\square$

Now for the original proposition. We will prove this by induction. First we prove the base case, where $n = 1$. Then, $1^3 + 5 \cdot 1 = 6$, which is definitely divisible by 6.

For the inductive step, we assume that the proposition holds for a certain $m \in \mathbb{N}$. So, $m^3 + 5m$ is divisible by 6. When we increase m by 1, we get: $(m + 1)^3 + 5(m + 1)$. Expanded, this is the same as $m^3 + 3m^2 + 8m + 6$. When we rearrange the terms we can get the following expression: $(m^3 + 5m) + (3m^2 + 3m + 6)$. From Lemma 4.1, we know that the latter part is divisible by 6. The prior part is divisible by 6 because of the assumption of the inductive step. So together, this expression is also divisible by 6. $\square$

Exercise 0.3.19

Give an example of a countably infinite collection of finite sets $A_1, A_2, \dots$, whose union is not a finite set.

The easiest example is simply the collection of singleton sets containing a natural number. So each set $A_i := \{i\} \forall i \in \mathbb{N}$. Since $\mathbb{N}$ is countably infinite, so the collection of sets. Each set is definitely finite, because they all contain just one element. Finally, the union of the collection of sets is equal to $\mathbb{N}$, which is not a finite set.

Exercise 6