Week 1

Exercise 0.3.6

Prove:
a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
b) $A \cup (B \cap C) = (A \cup B) \cap (a \cup C)$

a) In order to prove this equivalence, we have to prove the implication both ways. We use two lemmas for this.

Lemma 1.1 — $A \cap (B \cup C) \implies (A \cap B) \cup (A \cap C)$

Let $x \in A \cap (B \cup C)$. By the definition of set intersection, $x \in A$ and $x \in B \cup C$. By the definition of set union, $x \in A$ and ($x \in B$ or $x \in C$). From propositional logic we know that for propositions P, Q and R the following holds: $P \land (Q \lor R) \iff (P \land Q) \lor (P \land R)$. So, substituting for this particular case yields ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). Using the definition of set intersection again gets $x \in A \cap B$ or $x \in A \cap C$. Using the definition of set union again gives $x \in (A \cap B) \cup (A \cap C)$. □

Lemma 1.2 — $(A \cap B) \cup (A \cap C) \implies A \cap (B \cup C)$

Let $x \in (A \cap B) \cup (A \cap C)$. By the definition of set union, $x \in (A \cap B)$ or $x \in (A \cap C)$. By the definition of set intersection, ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in B$). Using the same propositional logical equivalence as in Lemma 1.1, this gives $x \in A$ and ($x \in B$ or $x \in C$). Wrapping up, we use the definition of set union to get $x \in A$ and $x \in B \cup C$ and the definition of intersection to get $x \in A \cap (B \cup C)$. □

Using Lemma 1.1 and 1.2, we get the desired equivalence of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

b) This proof is so similar to a) that it feels like a waste of time and will therefore be left to the reader.

Exercise 0.3.11

Prove by induction that $n < 2^n$ for all $n \in \mathbb{N}$.

For this proof we will use induction. For this, we have to prove the base case, i.e. $n = 1$, and the inductive step, $n < 2^n \implies n + 1 < 2^{n+1}$.

First, let's prove the base case. When $n = 1$, we get $1 < 2^1$, which is certainly true.

Then, for the inductive step. We assume that the proposition holds for any $m \in \mathbb{N}$. So, $m < 2^m$. Multiplying both sides with 2 gives $2m < 2^{m+1}$. Since $m \geq 1$, $m + 1 < 2m$, and thus $m + 1 < 2m < 2^{m+1}$.

Since both the base case and inductive step hold, we can close the induction, proving the proposition. □

Exercise 0.3.12

Show that for a finite set A of cardinality n, the cardinality of $\mathcal{P}(A)$ is 2^n.

The power set of a set A, $\mathcal{P}(A)$, is defined as the set of all possible subsets of A. This is very similar to an inclusion/exclusion problem. It is built up by all the possible combinations of the different elements being either inside a certain subset or not. For all possible subsets of A, we have that for every element $x \in A$ there are 2 possibilities, either x is in the subset or it isn't. This means that for every additional element, the number of subsets increases by a factor of 2, with a minimum of 1, in case of $A = \emptyset$. We will prove this formally now, using induction.

For this, the base case is a set of 1 element (but the theorem also holds for the empty set, where $n = 0$). Let us assume that $A := \{\pi\}$. Then the cardinality of $\mathcal{P}(A)$ is 2^1, with $\mathcal{P}(A) = \{\emptyset, \{\pi\}\}$.

For the inductive step, we assume that for any set B of cardinality m, the cardinality of the power set of B is 2^m. Then, we will add an element $x \notin B$ to B to increase its cardinality by 1, to $m + 1$, creating a new set C. Note that all the possible subsets of B are still viable subsets of C, since $B \subset C$. In order to create the new subsets, we can simply keep all the subsets of B, duplicate them and take the union with the new element x, so now we also have all combinations of the old sets with possibly x being in them. Since this doubles the number of subsets, the cardinality of $\mathcal{P}(C)$ is 2^{m+1}.

Both the base case and the inductive step hold, which closes the induction and proves the proposition. □

Exercise 0.3.15

Prove that $n^3 + 5n$ is divisible by 6 for all $n \in \mathbb{N}$.

In order to prove this proposition, we will use induction. To do this, we need to prove the following lemma, of which we will see the usefulness later:

Lemma 4.1 — $3n^2 + 3n + 6$ *is divisible by 6 for all* $n \in \mathbb{N}$.

This lemma we will also prove by induction. For this, we prove the base case and the inductive step. First, for the base case we have $n = 1$, yielding $3 \cdot 1^2 + 3 \cdot 1 + 6 = 12$, which is divisibly by 6.

Then, for the inductive step we assume that the lemma holds for a certain $m \in \mathbb{N}$. So, $3m^2 + 3m + 6$ is divisible by 6. Substituting m with $m + 1$ gives $3(m + 1)^2 + 3(m + 1) + 6$, which can be expanded to $3m^2 + 9m + 12$. Rewriting this with our assumption in mind gives the following: $(3m^2 + 3m + 6) + (6m + 6)$. We know from our assumption that the first part is divisible by 6, and since $m \in \mathbb{N}$, $6m + 6$ is also divisible by 6, and so the whole expression is as well. □

Now for the original proposition. We will prove this by induction. First we prove the base case, where $n = 1$. Then, $1^3 + 5 \cdot 1 = 6$, which is definitely divisible by 6.

For the inductive step, we assume that the proposition holds for a certain $m \in \mathbb{N}$. So, $m^3 + 5m$ is divisible by 6. When we increase m by 1, we get: $(m + 1)^3 + 5(m + 1)$. Expanded, this is the same as $m^3 + 3m^2 + 8m + 6$. When we rearrange the terms we can get the following expression: $(m^3 + 5m) + (3m^2 + 3m + 6)$. From Lemma 4.1, we know that the latter part is divisible by 6. The prior part is divisible by 6 because of the assumption of the inductive step. So together, this expression is also divisible by 6. □

Exercise 0.3.19

Give an example of a countably infinite collection of finite sets $A_1, A_2, ...,$ whose union is not a finite set.

The easiest example is simply the collection of singleton sets containing a natural number. So each set $A_i := \{i\} \forall i \in \mathbb{N}$. Since $\mathbb{N}$ is countably infinite, so the collection of sets. Each set is definitely finite, because they all contain just one element. Finally, the union of the collection of sets is equal to $\mathbb{N}$, which is not a finite set.

Exercise 6

a) Compute $f(4/15)$. Find q such that $f(q) = 108$.
b) Use the **Theorem** to prove that f is a bijection.

See the assignment PDF for the full assignment specification and theorem.

a) $\frac{4}{15}$, if written as a product of prime factors, is equal to $\frac{2^2}{3^1 \cdot 5^1}$. Since this fraction is not a natural number, we have to use the second part of the definition of f. So, $f(q) = 2^{2 \cdot 2} \cdot 3^{2 \cdot 1 - 1} \cdot 5^{2 \cdot 1 - 1} = 240$.

For the inverse of f, it is still necessary to compute the factorization in prime numbers. Using the powers of the primes we can deduce whether the prime present is, if applicable, part of either the numerator or the denominator. $180 = 2^2 \cdot 3^2 \cdot 5^1$. Because of the way f is defined, we know that all the prime factors with an even power are part of the numerator and all prime factors with an odd power are part of the denominator (except 1, which just maps to itself). When we backtrack using this information, we then get the following fraction: $\frac{2^1 \cdot 3^1}{5^1} = \frac{6}{5}$.

b) In order to prove that f is a bijection, we have two prove that f is injective and surjective.

Surjectivity

Injectivity