Part I

Assignment 4

Exercise 1

We say a set $F \subset \mathbb{R}$ is *closed* if its complement $F^c := \mathbb{R} \backslash F$ is open. Since $\emptyset$ and $\mathbb{R}$ are open, it follows that $\emptyset$ and $\mathbb{R}$ are closed as well.

(a) Let $a, b \in \mathbb{R}$ with $a < b$. Prove that $[a, b]$ is closed.

(b) Is the set $\mathbb{Z} \subset \mathbb{R}$ closed? Provide a proof to substantiate your claim.

(c) Is the set of rationals $\mathbb{Q} \subset \mathbb{R}$ closed? Provide a proof to substantiate your claim.

(a) The complement of $[a, b]$ is equal to the union of $(-\infty, a)$ and (b, ∞). So, we have to prove that both these sets are open. But we have already done so in the assignment of the previous week, so I'll just leave it at that.

(b) To prove that $\mathbb{Z} \subset \mathbb{R}$ is closed, we have to prove that the complement is open. Since the complement consists of the union of a countably infinite number of open intervals (a, b) such that $a < b$, we know from a combination of earlier exercises that this is the case. This is because any interval (a, b) is open if $a, b \in \mathbb{R}$ such that $a < b$ and for any two open interval A and B that are open, then $A \cup B$ is open as well.

(c) I claim that the set of rationals isn't closed in $\mathbb{R}$. This is because there doesn't exist any interval (a, b) where $a, b \in \mathbb{Q}$ such that $a < b$, since for any a and b you can always find a c such that $a < c < b$. This makes it impossible to find an $\varepsilon > 0$ such that for any $x \in (a, b)$, $(x - \varepsilon, x + \varepsilon)$ is also in (a, b) but in such a way that it only contains irrational numbers. This argument makes use of the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 2

(a) Let Λ be a set (not necessarily a subset of $\mathbb{R}$), and for each $\lambda \in \Lambda$, let $F_\lambda \in \mathbb{R}$. Prove that if F_λ is closed for all $\lambda \in \Lambda$ then the set

$$\bigcap_{\lambda \in \Lambda} F_\lambda = \{x \in \mathbb{R} : x \in F_\lambda \text{ for all } \lambda \in \Lambda\}$$

is closed.

(b) Let $n \in \mathbb{N}$, and let $F_1, ..., F_n \subset \mathbb{R}$. Prove that if $F_1, ..., F_n$ are closed then the set $\bigcup_{m=1}^{n} F_m$ is closed.

This exercise is very similar to an exercise of the previous assignment, in which we looked at unions and intersections of *open* intervals, whereas in this exercise, it's all about closed intervals. As the definition of closed intervals is intricately linked to the definition of open intervals, the following arguments will look very similar and shouldn't be surprising.

(a)

(b)