Part I

Assignment 3

Exercise 1

Suppose $x, y \in \mathbb{R}$ and $x < y$. Prove that there exists $i \in \mathbb{R} \backslash \mathbb{Q}$ such that $x < i < y$.

If either x or y (or both) are not rational numbers, we can simply take the average like so: $\frac{x+y}{2}$, in a similar way we did for the rationals. Since x or y isn't rational, the resulting fraction will also not be a rational and this proves the statement.

Now if $x, y \in \mathbb{Q}$, we cannot use this average trick, because the resulting fraction will be a rational itself and so it doesn't satisfy the restriction that it must be in $\mathbb{R} \backslash \mathbb{Q}$. So we have to take a different approach.

Let $x, y \in \mathbb{Q}$ with $x < y$ and $m := \frac{x+y}{2}$, so $x < m < y$. Then, let $X = \{a \in \mathbb{R} : x < a < m\}$ and let $Y = \{b \in \mathbb{R} : m < b < y\}$. Since $x < m$ and $m < y$, these are nonempty and they are bounded, because of the restrictions $x < a < m$ and $m < b < y$. So, there exists $k \in X$, that is not rational such that $x < k < m$ and there exists $h \in Y$ that is not rational such that $m < h < y$. Pick either k or h as i, since $x < k < m < h < y$. □

Exercise 2

Let $E \subset (0, 1)$ be the set of all real numbers with decimal representation using only the digits 1 and 2:

$$E := \{x \in (0, 1) : \forall j \in \mathbb{N}, \exists d_j \in \{1, 2\} \text{ such that } x = 0.d_1 d_2 ...\} \tag{2.1}$$

Prove that $|E| = |\mathcal{P}(\mathbb{N})|$.