Week 2

Exercise 1.1.1

> Let F be an ordered field and $x, y, z \in F$. If $x < 0$ and $y < z$, then $xy > xz$.

So let's assume the premise. F is an ordered field and $x, y, z \in F$, and we choose x, y and z such that $x < 0$ and $y < z$.

From $x < 0$ it follows that $(-x) > 0$. From $y < z$ it follows that $0 < z - y$. From both of these, we can conclude that $0 < (-x)(z - y)$. Working out the right side with the distributive law, gives $0 < (-x \cdot z) - (-x \cdot y)$. Using $-1 \cdot -1 = 1$, gives $0 < (-xz) - (-xy)$, thus $0 < xy - xz$. The right part can be split again: $xz < xy$. Then, the < can be flipped, which gives $xy > xz$. □

Exercise 1.1.2

> Let S be an ordered set. Let $A \subset S$ be a non-empty finite subset. Then A is bounded. Furthermore, $\inf A$ exists and in in A and $\sup A$ exists and is in A.

In order to prove that A is bounded, we have to prove that it has an upper and a lower bound. Let us prove that A is bounded above first.

In particular, we have to prove that $\exists a \in A$ such that $x \leq b$ for all $x \in E$. Since A is non-empty and finite, we can use induction on the cardinality of A, since that will always be some natural number n. So, we have to prove two cases: the base case, where $|A| = 1$, and the inductive step, where we will assume that when A has an upper bound when it has cardinality m, then it also has an upper bound when its cardinality is equal to $m + 1$.

The base case is quite simple; if $A = \{x\}$, then x is the greatest element and A has an upper bound. Now for the inductive step. We assume that for some set $B \subset S$ with cardinality m, B is bounded above. Thus, there is some $b \in B$ such that b is greater than all other elements in B. Now, let's add a new element $h \in S$ to B, such that h is distinct from all elements already in B and the cardinality of B is now $m + 1$. Then, since S is well ordered, we can compare h also to b. Either h is greater than this b, in which case h is the new greatest element, or it is less than b, in which case b stays the greatest element of B. In both cases however, B remains bounded above. □

A similar argument can be made to prove the existence of the lower bound, the supremum of A in A and the infimum of A in A [1]. This will be left to the reader.

Exercise 1.1.5

> Let S be an ordered set. Let $A \subset S$ and suppose b is an upper bound for A. Suppose $b \in A$. Show that $b = \sup A$.

[1] It might even be that I have already proven that A has a supremum present in A. Then that's also good enough to show that A is bounded, since in order for A to have a supremum, it must also be bounded.

So, let S be an ordered set, with $A \subset S$ and $b \in A$ being an upper bound for A. Since b is an upper bound, $a \leq b$ for all $a \in A$. Since $b \in A$ as well, we know that there is some element in A which is the greatest element of them all, and all other elements are smaller.

Now let's assume that $b \neq \sup A$. Then either some other element of A is the supremum, which would imply that b is not larger than this element, which is a contradiction. The other possibility is that there is an element $c \in S \backslash A$ that is the supremum. Because S is ordered, c must either be greater than, smaller than or equal to b. If $c < b$, c is not an upper bound of A and thus also not its supremum. If $c > b$, then b is an upper bound that is smaller than c and therefore c cannot be the supremum. The only option left is that $c = b$, and therefore $b = \sup A$. $\square$

Exercise 1.1.6

> Let S be an ordered set. Let $A \subset S$ be nonempty and bounded above. Suppose $\sup A$ exists and $\sup A \notin A$. Show that A contains a countably infinite subset.

Let S be an ordered set, with $A \subset S$ nonempty and bounded above. We assume that $b = \sup A$ exists and $b \notin A$ ($\implies b \in S \backslash A$). We are asked to show this then implies that $\exists X \subset A$ such that $|X| \geq |\mathbb{N}|$. We will prove this with a proof by contradiction.

We assume that no such set X exists, i.e. $|X| < |\mathbb{N}|$. So, A also doesn't have to countably infinite anymore. Since $b \notin A$ and A is ordered, finite and nonempty, there is a greatest element $a \in A$ such that $a < b$ and $x < a \; \forall x \in A$. So b is in fact not the least upper bound of A, which is in contradiction with our assumption earlier. Ergo, $|X| \geq |\mathbb{N}|$. Since X then is at least of the same cardinality as $\mathbb{N}$, it must also contain a countably infinite subset. $\square$

Exercise 1.2.7

> Prove the arithmetic-geometric mean inequality. That is, for two positive real numbers x, y, we have
> $$\sqrt{xy} \leq \frac{x+y}{2}.$$
> Furthermore, equality occurs if and only if $x = y$.

Let us prove the first statement first. So we let $x, y \in \mathbb{R}$ such that $x, y > 0$. Then we will prove the statement by contradiction. Hence, we assume that

$$\sqrt{xy} > \frac{x+y}{2}.$$

We can multiply both sides with 2. This results in $2\sqrt{xy} > x + y$. We can pull the left part into the right, so we get $0 > x - 2\sqrt{xy} + y$. We can restructure the right side to $0 > (\sqrt{x} - \sqrt{y})^2$. We know that $0 \leq z^2, \forall z \in \mathbb{R}$, so this is a contradiction. $\square$

Now to prove the second statement. We assume $x = y$ is a positive real number. Then,

$$\sqrt{xy} = \sqrt{x^2} = x = \frac{2x}{2} = \frac{x+y}{2}. \quad \square$$

Exercise 1.2.9

Let A and B be two nonempty bounded sets of real numbers. Define the set $C := \{a+b : a \in A, b \in B\}$. Show that C is a bounded set and that

$$\sup C = \sup A + \sup B \text{ and}$$
$$\inf C = \inf A + \inf B.$$

First, let us show that C is a bounded set. Since A and B are both subsets of $\mathbb{R}$, which is an ordered field, all elements of C must also be real numbers. Let a be an upper bound for A and b be an upper bound for B. So $x \leq a \; \forall x \in A$ and $y \leq b \; \forall y \in B$. Since C is defined as the sum of any element in A with any element in B, an upper bound of C, c, can be found as $c \leq a + b$. A similar argument can be made for the lower bound of C, which makes C bounded. □

To prove that $\sup C = \sup A + \sup B$, we will show that $\sup C \geq \sup A + \sup B$ and $\sup C \leq \sup A + \sup B$.

Let $a = \sup A$ and $b = \sup B$. So $x \leq a$ for all $x \in A$ and $y \leq b$ for all $y \in B$. Then, $x + y \leq a + b$. Since $z \leq x + y$ for all $z \in C$ because of the definition of C, $z \leq a + b$. In other words, $\sup C \leq \sup A + \sup B$.

Now to prove the other direction. Let $c = \sup C$. So $z \leq c$ for all $c \in C$. Since all elements in C are the sum of an element $x \in A$ and $y \in B$, $x + y \leq c$ for all x, y. The least upper bound for these x and y can be given by the supremum; $x \leq \sup A$ and $y \leq \sup B$. So, $\sup A + \sup B \leq c \implies \sup A + \sup B \leq \sup C$, completing the equality. □

A similar argument can be given for the infimum, which is left to the reader.

Exercise 7

Let
$$E = \{x \in \mathbb{R} : x > 0 \text{ and } x^3 < 2\}.$$

a) Prove that E is bounded above.
b) Let $r = \sup E$ (which exists by part a)). Prove that $r > 0$ and $r^3 = 2$.
Hint: Adapt the proof used in Example 1.2.3.

So, let E and r be defined as in the exercise statement. Then:

a) $x \leq 2$ is an upper bound for E, as $2 \cdot 2 \cdot 2 = 8$. So, E is bounded above.

b) As $1 \in E$, $r \geq 1 > 0$, so the first part of the statement holds. In order to show that $r^3 = 2$, we want to show that $r^3 \leq 2$ and $r^3 \geq 2$ hold.

First, let's show that $r^3 \geq 2$. We will take a similar approach as in Example 1.2.3 from the textbook. So, take a positive number s such that $s^3 < 2$. We wish to find an $h > 0$ such that $(s + h)^3 < 2$. As $2 - s^3 > 0$, we have $\frac{2-s^3}{3s^2+3s+1} > 0$. Choose an $h \in \mathbb{R}$ such that $0 < h < \frac{2-s^3}{3s^2+3s+1}$. Furthermore, assume $h < 1$. Estimate,

$$\begin{aligned}(s+h)^3 - s^3 &= h(3s^2 + 3sh + h^2) \\ &< h(3s^2 + 3s + 1) && \text{(since } h < 1) \\ &< 2 - s^3 && \text{(since } h < \frac{2 - s^3}{3s^2 + 3s + 1}).\end{aligned}$$

Therefore, $(s+h)^3 < 2$. Hence $s+h \in E$, but as $h > 0$, we have $s+h > s$. So $s < r = \sup E$. As s was an arbitrary positive number such that $s^3 < 2$, it follows that $r^3 \geq 2$.

Now take an arbitrary positive number s such that $s^3 > 2$. We wish to find an $h > 0$ such that $(s-h)^3 > 2$ and $s-h$ is still positive. As $s^3 - 2 > 0$, we have that $\frac{s^3-2}{3s^2+1} > 0$. Let $h := \frac{s^3-2}{3s^2+1}$, and check that $s - h = s - \frac{s^3-2}{3s^2+1} = \frac{2s^3+s+2}{3s^2+1} > 0$. Assume that $h < 1$. Estimate,

$$\begin{aligned}s^3 - (s-h)^3 &= h(3s^2 - 3sh + h^2) \\ &< h(3s^2 + h^2) && \text{(since } s > 0 \text{ and } h > 0) \\ &< h(3s^2 + 1) && \text{(since } h < 1) \\ &= s^3 - 2 && \text{(because of the definition of } h).\end{aligned}$$

By subtracting s^3 from both sides and multiplying by -1, we find $(s-h)^3 > 2$. Therefore, $s - h \notin E$. Moreover, if $x \geq s - h$, then $x^3 \geq (s-h)^3 > 2$ (as $x > 0$ and $s - h > 0$) and so $x \notin E$. Thus, $s - h$ is an upper bound for E. However, $s - h < s$, or in other words, $s > r = \sup E$. Hence, $r^3 \leq 2$.

Together, $r^3 \geq 2$ and $r^3 \leq 2$ imply $r^3 = 3$. □