Week 2

Exercise 1.1.1

Let F be an ordered field and $x, y, z \in F$. If $x < 0$ and $y < z$, then $xy > xz$.

So let's assume the premise. F is an ordered field and $x, y, z \in F$, and we choose x, y and z such that $x < 0$ and $y < z$.

From $x < 0$ it follows that $(-x) > 0$. From $y < z$ it follows that $0 < z - y$. From both of these, we can conclude that $0 < (-x)(z - y)$. Working out the right side with the distributive law, gives $0 < (-x \cdot z) - (-x \cdot y)$. Using $-1 \cdot -1 = 1$, gives $0 < (-xz) - (-xy)$, thus $0 < xy - xz$. The right part can be split again: $xz < xy$. Then, the < can be flipped, which gives $xy > xz$. $\square$

Exercise 1.1.2

Let S be an ordered set. Let $A \subset S$ be a non-empty finite subset. Then A is bounded. Furthermore, $\inf A$ exists and in in A and $\sup A$ exists and is in A.

In order to prove that A is bounded, we have to prove that it has an upper and a lower bound. Let us prove that A is bounded above first. In particular, we have to prove that $\exists a \in A$ such that $x \leq b$ for all $x \in E$. Since A is non-empty and finite, we can use induction on the cardinality of A, since that will always be some natural number n. So, we have to prove the two case: the base case, where $|A| = 1$, and the inductive step, where we will assume that when A has an upper bound when it has cardinality m, it also has an upper bound when its carindality is equal to $m + 1$.