Week 1

Exercise 0.3.6

Prove:

a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

b) $A \cup (B \cap C) = (A \cup B) \cap (a \cup C)$

a) In order to prove this equivalence, we have to prove the implication both ways. We use two lemmas for this.

Lemma 1.1 — $A \cap (B \cup C) \implies (A \cap B) \cup (A \cap C)$

Let $x \in A \cap (B \cup C)$. By the definition of set intersection, $x \in A$ and $x \in B \cup C$. By the definition of set union, $x \in A$ and ($x \in B$ or $x \in C$). From propositional logic we know that for propositions P, Q and R the following holds: $P \wedge (Q \vee R) \iff (P \wedge Q) \vee (P \wedge R)$. So, substituting for this particular case yields ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). Using the definition of set intersection again gets $x \in A \cap B$ or $x \in A \cap C$. Using the definition of set union again gives $x \in (A \cap B) \cup (A \cap C)$. □

Lemma 1.2 — $(A \cap B) \cup (A \cap C) \implies A \cap (B \cup C)$

Let $x \in (A \cap B) \cup (A \cap C)$. By the definition of set union, $x \in (A \cap B)$ or $x \in (A \cap C)$. By the definition of set intersection, ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in B$). Using the same propositional logical equivalence as in Lemma 1.1, this gives $x \in A$ and ($x \in B$ or $x \in C$). Wrapping up, we use the definition of set union to get $x \in A$ and $x \in B \cup C$ and the definition of intersection to get $x \in A \cap (B \cup C)$. □

Using Lemma 1.1 and 1.2, we get the desired equivalence of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

b) This proof is so similar to a) that it feels like a waste of time and will therefore be left to the reader.

Exercise 0.3.11

> Prove by induction that $n < 2^n$ for all $n \in \mathbb{N}$.

For this proof we will use induction. For this, we have to prove the base case, i.e. $n = 1$, and the inductive step, $n < 2^n \implies n + 1 < 2^{n+1}$.

First, let's prove the base case. When $n = 1$, we get $1 < 2^1$, which is certainly true.

Then, for the inductive step. We assume that the proposition holds for any $m \in \mathbb{N}$. So, $m < 2^m$. Multiplying both sides with 2 gives $2m < 2^{m+1}$. Since $m \geq 1$, $m + 1 < 2m$, and thus $m + 1 < 2m < 2^{m+1}$.

Since both the base case and inductive step hold, we can close the induction, proving the proposition. □

Exercise 0.3.12

> Show that for a finite set A of cardinality n, the cardinality of $\mathcal{P}(A)$ is 2^n.

The power set of a set A, $\mathcal{P}(A)$, is defined as the set of all possible subsets of A. This is very similar to an inclusion/exclusion problem. It is built up by all the possible combinations of the different elements being either inside a certain subset or not. For all possible subsets of A, we have that for every element $x \in A$ there are 2 possibilities, either x is in the subset or it isn't. This means that for every additional element, the number of subsets increases by a factor of 2, with a minimum of 1, in case of $A = \emptyset$. We will prove this formally now, using induction.

For this, the base case is a set of 1 element (but the theorem also holds for the empty set, where $n = 0$). Let us assume that $A := \{\pi\}$. Then the cardinality of $\mathcal{P}(A)$ is 2^1, with $\mathcal{P}(A) = \{\emptyset, \{\pi\}\}$.

For the inductive step, we assume that for any set B of cardinality m, the cardinality of the power set of B is 2^m. Then, we will add an element $x \notin B$ to B to increase its cardinality by 1, to $m + 1$, creating a new set C. Note that all the possible subsets of B are still viable subsets of C, since $B \subset C$. In order to create the new subsets, we can simply keep all the subsets of B, duplicate them and take the union with the new element x, so now we also have all combinations of the old sets with possibly x being in them. Since this doubles the number of subsets, the cardinality of $\mathcal{P}(C)$ is 2^{m+1}.

Both the base case and the inductive step hold, which closes the induction and proves the proposition. □