Part I

Assignment 3

Exercise 1

> Suppose $x, y \in \mathbb{R}$ and $x < y$. Prove that there exists $i \in \mathbb{R}\backslash\mathbb{Q}$ such that $x < i < y$.

If either x or y (or both) are not rational numbers, we can simply take the average like so: $\frac{x+y}{2}$, in a similar way we did for the rationals. Since x or y isn't rational, the resulting fraction will also not be a rational and this proves the statement.

Now if $x, y \in \mathbb{Q}$, we cannot use this average trick, because the resulting fraction will be a rational itself and so it doesn't satisfy the restriction that it must be in $\mathbb{R}\backslash\mathbb{Q}$. So we have to take a different approach.

Let $x, y \in \mathbb{Q}$ with $x < y$ and $m := \frac{x+y}{2}$, so $x < m < y$. Then, let $X = \{a \in \mathbb{R} : x < a < m\}$ and let $Y = \{b \in \mathbb{R} : m < b < y\}$. Since $x < m$ and $m < y$, these are nonempty and they are bounded, because of the restrictions $x < a < m$ and $m < b < y$. So, there exists $k \in X$, that is not rational such that $x < k < m$ and there exists $h \in Y$ that is not rational such that $m < h < y$. Pick either k or h as i, since $x < k < m < h < y$. $\qquad\square$

Exercise 2

> Let $E \subset (0, 1)$ be the set of all real numbers with decimal representation using only the digits 1 and 2:
>
> $$E := \{x \in (0, 1) : \forall j \in \mathbb{N}, \exists d_j \in \{1, 2\} \text{ such that } x = 0.d_1 d_2...\}$$
>
> Prove that $|E| = |\mathcal{P}(\mathbb{N})|$.

As a hint to this exercise: *Consider the function $f : E \rightarrow \mathcal{P}(\mathbb{N})$ such that if $x \in E, x = 0.d_1 d_2...$,*

$$f(x) = \{j \in \mathbb{N} : d_j = 2\}.$$

In order to prove that 2 sets are of equal cardinality, we need to prove that there is a bijective function between the 2 sets. In this case, the aforementioned hint function does the trick. Non-formally speaking, it is exactly what we are looking for: it is a (weird) representation of the power set of natural numbers, in that for every decimal, represented by a natural number, it is decided if that decimal is a 2 or a 1. This is similar to the actual power set of the natural numbers, in which for every natural number it is decided whether the number is in a subset or not.

Now for a formal proof. To show that f is bijective, we need to show that it is surjective and injective.

Injectivity of f In order to show that f is injective, we have to show that for every $x \in E$, there is a unique $y \in \mathcal{P}(\mathbb{N})$ for the function, by showing that $f(a) = f(b) \implies a = b$.

So, let's assume that for some $a, b \in E$, $f(a) = f(b)$. So, there two sets of natural numbers $\{a_1, a_2, ..., a_n\} = \{b_1, b_2, ..., b_m\}$. Equality in sets means that every element that is present in the one set, is present in the other, and vice versa. No element that is present in either set, is missing in the other. So, in this case, both sets will represent the same sequence of digits that are 2. Because the only other option for digits is 1, that means the complete digital representation of a and b are known, unique and the same. This concludes the proof for injectivity.

Surjectivity of f To prove surjectivity, we need to prove that for any arbitrary $y \in \mathcal{P}(\mathbb{N})$, there exists a corresponding $x \in E$ such that $f(x) = y$.

So, take an arbitrary $y = \{y_1, y_2, ..., y_n\}$, where each $y_i \in \mathbb{N}$ and thus $y \in \mathcal{P}(\mathbb{N})$. Then, corresponding $x \in E$ can be constructed easily as follows. Take a decimal number $0.d_1 d_2 ...$ and turn every decimal d_i for which $i \in y$ into a 2, and every other decimal into a 1. Since every decimal can only be a 1 or 2, this handles every decimal correctly. Also, $f(x)$ will be in $\mathcal{P}(\mathbb{N})$.

Since, f is 1-to-1 and onto, f is bijective. Then, because there exists a bijective function from E to $\mathcal{P}(\mathbb{N})$, $|E| = |\mathcal{P}(\mathbb{N})|$. □

Exercise 3

> *(a)* Let A and B be two disjoint, countably infinite sets. Prove that $A \cup B$ is countably infinite.
> *(b)* Prove that the set of irrational numbers, $\mathbb{R} \backslash \mathbb{Q}$, is uncountable. You may use the facts discussed in the lectures that $\mathbb{R} \backslash \mathbb{Q}$ is infinite and $\mathbb{R}$ is uncountable without proof.

(a) So let A and B be two disjoint, countably infinite sets. Since these sets are countably infinite, a bijective function to $\mathbb{N}$ exists for both functions separately. It is then straightforward to map both these function together to $\mathbb{Z}$ instead, in the following way. Let f be the bijective function such that $f : A \to \mathbb{N}$ and let g be the bijective function such that $g : B \to \mathbb{N}$. Then, we can define a new function $h : A \cup B \to \mathbb{Z}$ as

$$h(x) = f(x) \text{ if } x \in A$$
$$= -g(x) \text{ if } x \in B.$$

Since $A \cap B = \varnothing$, this function is unambiguously defined. Since $\mathbb{Z}$ is countably infinite, $A \cup B$ is countably infinite as well. □

(b) Because of part (a), we know that if we have two disjoint, countably infinite sets and join them, the result is still countably infinite. The opposite must then also be true: if we have a countably infinite set and we divide it into two disjoint subsets, both of which are infinite, then they still must be countable.

So then, for $\mathbb{R} \backslash \mathbb{Q}$, we know that $\mathbb{R}$ is uncountable infinite. So when we split it into rational and irrational subsets, from which we know that $\mathbb{Q}$ is countably infinite, $\mathbb{R} \backslash \mathbb{Q}$ must be at least and at most uncountably infinite. □

Exercise 4

Let A be a subset of $\mathbb{R}$ which is bounded above, and let a_0 be an upper bound for A. Prove that $a_0 = \sup A$ if and only if for every $\varepsilon > 0$, there exists $a \in A$ such that $a_0 - \varepsilon < a$.

Let $A \subset \mathbb{R}$, with A bounded above by a_0. So, we have to prove the implication both ways. First, let's prove that the implication to the right ($\rightarrow$).

Assume that $a_0 = \sup A$, so for all $a \in A$, $a \leq a_0$. Also, let $\varepsilon > 0$. If $a_0 \in A$, then we pick a_0 as a and get $a_0 - \varepsilon < a_0$, which holds $\forall \varepsilon > 0$. If $a_0 \notin A$, then we choose a as the average of a_0 and $a_0 - \varepsilon$, which is definitely smaller than a_0. We are allowed to pick this as a, because we assume without loss of generality that $a \geq \inf A$. Then we get

$$
\begin{aligned}
a_0 - \varepsilon &< \frac{a_0 + a_0 - \varepsilon}{2} \\
&< a_0 - \frac{\varepsilon}{2} \implies \\
-\varepsilon &< -\frac{\varepsilon}{2}.
\end{aligned}
$$

Since $\varepsilon > 0$, this always holds.

Now for the implication to the left ($\leftarrow$).

Assume now that the right side is true, i.e. let's assume that $\forall \varepsilon > 0$, there exists $a \in A$ such that $a_0 - \varepsilon < a$. Again, let us first investigate the case where $a_0 \in A$. Well certainly still, if a_0 is an upper bound for A and it is also part of the set itself, it must be the supremum. Then, let's assume that $a_0 \notin A$. Now, for all positive ε, we know there exists an $a \in A$ such that $a \neq a_0$ and $a_0 - \varepsilon < a$. So, $a_0 < a + \varepsilon$. a_0 is an upper bound and we can find a's such that $a + \varepsilon$ is always bigger than a_0, so a_0 must be the supremum. If a_0 wasn't the supremum, then there must be some b such that $a < b < a_0 < a + \varepsilon$. WIP