Week 2

Exercise 1.1.1

> Let F be an ordered field and $x, y, z \in F$. If $x < 0$ and $y < z$, then $xy > xz$.

So let's assume the premise. F is an ordered field and $x, y, z \in F$, and we choose x, y and z such that $x < 0$ and $y < z$.

From $x < 0$ it follows that $(-x) > 0$. From $y < z$ it follows that $0 < z - y$. From both of these, we can conclude that $0 < (-x)(z - y)$. Working out the right side with the distributive law, gives $0 < (-x \cdot z) - (-x \cdot y)$. Using $-1 \cdot -1 = 1$, gives $0 < (-xz) - (-xy)$, thus $0 < xy - xz$. The right part can be split again: $xz < xy$. Then, the < can be flipped, which gives $xy > xz$. □

Exercise 1.1.2

> Let S be an ordered set. Let $A \subset S$ be a non-empty finite subset. Then A is bounded. Furthermore, $\inf A$ exists and in in A and $\sup A$ exists and is in A.

In order to prove that A is bounded, we have to prove that it has an upper and a lower bound. Let us prove that A is bounded above first.

In particular, we have to prove that $\exists a \in A$ such that $x \leq b$ for all $x \in E$. Since A is non-empty and finite, we can use induction on the cardinality of A, since that will always be some natural number n. So, we have to prove two cases: the base case, where $|A| = 1$, and the inductive step, where we will assume that when A has an upper bound when it has cardinality m, then it also has an upper bound when its cardinality is equal to $m + 1$.

The base case is quite simple; if $A = \{x\}$, then x is the greatest element and A has an upper bound. Now for the inductive step. We assume that for some set $B \subset S$ with cardinality m, B is bounded above. Thus, there is some $b \in B$ such that b is greater than all other elements in B. Now, let's add a new element $h \in S$ to B, such that h is distinct from all elements already in B and the cardinality of B is now $m + 1$. Then, since S is well ordered, we can compare h also to b. Either h is greater than this b, in which case h is the new greatest element, or it is less than b, in which case b stays the greatest element of B. In both cases however, B remains bounded above. □

A similar argument can be made to prove the existence of the lower bound, the supremum of A in A and the infimum of A in A [1]. This will be left to the reader.

Exercise 1.1.5

> Let S be an ordered set. Let $A \subset S$ and suppose b is an upper bound for A. Suppose $b \in A$. Show that $b = \sup A$.

[1] It might even be that I have already proven that A has a supremum present in A. Then that's also good enough to show that A is bounded, since in order for A to have a supremum, it must also be bounded.